Learning Module 11: Yield-Based Bond Duration Measures and Properties

Fixed Income

Table of contents

Download Files

Download PDF | Download Word |

Modified Duration (ModDur)

Using Macaulay Duration (equation 2) from Learning module 10

Recall that the price, PV, of an option-free bond is the present value of the bond's cash flows.

$$PV = \frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + ... + \frac{PMT}{(1+r)^N} + \frac{FV}{(1+r)^N}$$

First, we take the derivative with respect to r, the yield-to-maturity:

$$\frac{dPV}{dr} = \frac{(-1)PMT}{(1+r)^2} + \frac{(-2)PMT}{(1+r)^3} + ... + \frac{(-N)PMT}{(1+r)^{N+1}} + \frac{(-N)FV}{(1+r)^{N+1}}$$

Then, we factor out $-\frac{1}{(1+r)}$ to get

$$\frac{dPV}{dr} = -\frac{1}{(1+r)}\left[\frac{(1)PMT}{(1+r)^1} + \frac{(2)PMT}{(1+r)^2} + ... + \frac{(N)PMT}{(1+r)^N} + \frac{(N)FV}{(1+r)^N}\right]$$

While this gives us the change in a bond's price for a change in yield, it does so in terms of PV, but percentage change in price would be more useful. To get percentage change, we divide by PV (price):

$$\frac{\frac{dPV}{dr}}{PV} = \frac{-\frac{1}{(1+r)}\left[\frac{(1)PMT}{(1+r)^1} + \frac{(2)PMT}{(1+r)^2} + ... + \frac{(N)PMT}{(1+r)^N} + \frac{(N)FV}{(1+r)^N}\right]}{PV}.$$

Look closely at the term in brackets. Each $\frac{PMT}{(1+r)^t}/PV$ is the present value of that cash flow expressed as a percentage of the bond price, which is then multiplied by the time to receipt of that cash flow. In other words, the term in brackets divided by PV is the Macaulay duration, $MacDur$, introduced in prior lessons. We can substitute $MacDur$ in the equation to obtain

$$\frac{\frac{dPV}{dr}}{PV} = -\frac{1}{(1+r)} \times \text{MacDur}$$

or

$$\frac{\frac{dPV}{dr}}{PV} = -\frac{MacDur}{(1+r)} \tag{1}$$

Without the negative sign, this is known as a bond's modified duration, or $ModDur$:

$$ModDur = \frac{MacDur}{(1+r)} \tag{2}$$

View Markdown Source

```
### Modified Duration (ModDur)

*Using Macaulay Duration (equation 2) from Learning module 10*

Recall that the price, $PV$, of an option-free bond is the present value of
the bond’s cash flows.

$$
PV = \frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + \ldots +
\frac{PMT}{(1+r)^N} + \frac{FV}{(1+r)^N}
$$

First, we take the derivative with respect to $r$, the yield-to-maturity:

$$
\frac{dPV}{dr}
=
\frac{(-1)PMT}{(1+r)^2}
+
\frac{(-2)PMT}{(1+r)^3}
+
\ldots
+
\frac{(-N)PMT}{(1+r)^{N+1}}
+
\frac{(-N)FV}{(1+r)^{N+1}}
$$

Then, we factor out $-\frac{1}{(1+r)}$ to get

$$
\frac{dPV}{dr}
=
-\frac{1}{(1+r)}
\left[
\frac{(1)PMT}{(1+r)^1}
+
\frac{(2)PMT}{(1+r)^2}
+
\ldots
+
\frac{(N)PMT}{(1+r)^N}
+
\frac{(N)FV}{(1+r)^N}
\right]
$$

While this gives us the change in a bond’s price for a change in yield, it
```

```
does so in terms of $PV$, but percentage change in price would be more useful.
To get percentage change, we divide by $PV$ (price):

$$
\frac{\frac{dPV}{dr}}{PV}
=
\frac{
-\frac{1}{(1+r)}
\left[
\frac{(1)PMT}{(1+r)^1}
+
\frac{(2)PMT}{(1+r)^2}
+
\ldots
+
\frac{(N)PMT}{(1+r)^N}
+
\frac{(N)FV}{(1+r)^N}
\right]
}{PV}.
$$

Look closely at the term in brackets. Each $\frac{PMT}{(1+r)^t}/PV$ is the
present value of that cash flow expressed as a percentage of the bond price,
which is then multiplied by the time to receipt of that cash flow. In other
words, the term in brackets divided by $PV$ is the Macaulay duration,
$MacDur$, introduced in prior lessons. We can substitute $MacDur$ in the
equation to obtain

$$
\frac{\frac{dPV}{dr}}{PV} = -\frac{1}{(1+r)} \times \text{MacDur}
$$

or

$$
\frac{\frac{dPV}{dr}}{PV} = -\frac{MacDur}{(1+r)} \tag{1}
$$

Without the negative sign, this is known as a bond’s modified duration, or
$ModDur$:

$$
ModDur = \frac{MacDur}{(1+r)} \tag{2}
$$
```

Estimate the percentage price change for a bond

$$
\%\Delta PV^{\text{Full}} \approx -\text{AnnModDur} \times \Delta \text{AnnYield} \tag{3}
$$

- Since $ModDur$ captures the relationship between a bond's price and its yield, we can use it to estimate the percentage price change for a bond given a change in its yield-to-maturity, if we substitute $-AnnModDur$ for the right side of Equation 1 and multiply both sides by dr, or the change in annualized yield-to-maturity

View Markdown Source

```
### Estimate the percentage price change for a bond

$$
\%\Delta PV^{\text{Full}} \approx -\text{AnnModDur} \times
\Delta \text{AnnYield} \tag{3}
$$


- Since $ModDur$ captures the relationship between a bond’s price and its
  yield, we can use it to estimate the percentage price change for a bond
  given a change in its yield-to-maturity, if we substitute $-AnnModDur$ for
  the right side of Equation 1 and multiply both sides by $dr$, or the change
  in annualized yield-to-maturity
```

Annualized Modified Duration

$$
AnnModDur \approx \frac{(PV_{-}) - (PV_{+})}{2 \times (\Delta Yield) \times (PV_0)} \tag{4}
$$

- PV_0 = The quoted full price of the bond
- $\Delta Yield$ = Changed in yield-to-maturity
- To estimate the slope, the yield-to-maturity is changed up and down by the same amount—the $\Delta Yield$—and is used to calculate corresponding bond prices PV_+ and PV_-. We can use these variables to find the slope of the line tangent to the price-yield curve: the difference between PV_+ and PV_- divided by twice the assumed change in the yield-to-maturity. To find the slope in terms of percentage change in PV_0, we further divide by PV_0. This is shown as Equation 4.

View Markdown Source

```
### Annualized Modified Duration

$$
AnnModDur \approx
\frac{(PV_{-}) - (PV_{+})}{2 \times (\Delta Yield) \times (PV_0)}
\tag{4}
$$

- $PV_0$ = The quoted full price of the bond
- $\Delta Yield$ = Changed in yield-to-maturity
- To estimate the slope, the yield-to-maturity is changed up and down by the
  same amount-the $\Delta Yield$-and is used to calculate corresponding bond
  prices $PV_{+}$ and $PV_{-}$. We can use these variables to find the slope
  of the line tangent to the price-yield curve: the difference between
  $PV_{+}$ and $PV_{-}$ divided by twice the assumed change in the
  yield-to-maturity. To find the slope in terms of percentage change in
  $PV_0$, we further divide by $PV_0$. This is shown as Equation 4.
```

Annualized Modified Duration

$$AnnModDur \approx AnnModDur \times (1 + r) \tag{5}$$

- The Macaulay duration also can be approximated by multiplying the approximate modified duration by 1 plus the yield per period.

View Markdown Source

```
### Annualized Modified Duration
$$
AnnModDur \approx AnnModDur \times (1+ r) \tag{5}
$$

- The Macaulay duration also can be approximated by multiplying the
  approximate modified duration by 1 plus the yield per period.
```

Money Duration (MoneyDur)

$$MoneyDur = AnnModDur \times PV^{Full}.$$

- $MoneyDur$ is the product of the annualized modified duration and the full price (PV^{Full}) of the bond, in either percent of par or the currency value of the position.

View Markdown Source

```
### Money Duration (MoneyDur)

$$
MoneyDur = AnnModDur \times PV^{Full}.
$$

- $MoneyDur$ is the product of the annualized modified
  duration and the full price ($PV^{Full}$) of the bond, in either percent
  of par or the currency value of the position.
```

Estimated change in the bond price in currency units

$$\Delta PV^{\text{Full}} \approx - \text{ MoneyDur} \times \Delta \text{Yield} \tag{7}$$

- The estimated change in the bond price in currency units is very similar to Equation 4. The difference is that for a given change in the annual yield-to-maturity ($\Delta Yield$), modified duration estimates the percentage price change while money duration estimates the change in currency units.

View Markdown Source

```
###  Estimated change in the bond price in currency units

$$
\Delta PV^{\text{Full}} \approx - \text{ MoneyDur} \times \Delta
\text{Yield} \tag{7}
$$

- The estimated change in the bond price in currency units is very similar to
  Equation 4. The difference is that for a given change in the annual
  yield-to-maturity ($\Delta Yield$), modified duration estimates the
  percentage price change while money duration estimates the change in
  currency units.
```

Price Value of a Basis Point (PVBP)

$$PVBP = \frac{(PV_{-}) - (PV_{+})}{2} \tag{8}$$

- $PVBP$ = an estimate of the change in the full price of a bond given a 1 bp change in its yield-to-maturity
- The $PVBP$ is also called the "$PV01$," standing for the "price value of an 01" or "present value of an 01," where "01" means 1 bp
- PV_{-} and PV_{+} are the full prices calculated by decreasing and increasing the yield-to-maturity by 1 bp
- The PVBP is particularly useful for bonds for which future cash flows are uncertain, such as callable bonds.

View Markdown Source

```
### Price Value of a Basis Point (PVBP)

$$
PVBP = \frac{(PV_{-}) - (PV_{+})}{2} \tag{8}
$$


- $PVBP$ = an estimate of the change in the full price of a bond given a 1 bp
  change in its yield-to-maturity
- The $PVBP$ is also called the "$PV01$," standing for the "price value of
  an 01" or "present value of an 01," where "01" means 1 bp
- $PV_{-}$ and $PV_{+}$ are the full prices calculated by decreasing and
  increasing the yield-to-maturity by 1 bp
- The PVBP is particularly useful for bonds for which future cash flows are
  uncertain, such as callable bonds.
```

Macaulay Duration: Non-callable perpetuities (MacDur)

$$
\text{MacDur} = \frac{(1+r)}{r} \tag{9}
$$

- A perpetuity or perpetual bond is a bond that does not mature, so there is no face or maturity value received at time T. The investor receives a fixed coupon payment forever unless the bond is called. Non-callable perpetuities are rare, but they have an interesting Macaulay duration

View Markdown Source

```
### Macaulay Duration: Non-callable perpetuities (MacDur)

$$
\text{MacDur} = \frac{(1+r)}{r} \tag{9}
$$

- A perpetuity or perpetual bond is a bond that does not mature, so there is
  no face or maturity value received at time T. The investor receives a fixed
  coupon payment forever unless the bond is called. Non-callable perpetuities
  are rare, but they have an interesting Macaulay duration
```

Macaulay duration for a floating-rate note or bond ($MACDur_{Floating}$)

$$
MacDur_{Floating} = \frac{(T-t)}{T} \tag{10}
$$

- As described in an earlier lesson, interest on floating-rate instruments varies depending on the level of a market reference rate (MRR) plus a quoted margin. At predetermined dates, payment amounts are reset to reflect changes in the MRR. Therefore, interest rate risk arises only between reset dates, because at the next reset date, coupon payments will adjust to the new MRR. Therefore, the Macaulay duration for a floating-rate note or bond is simply the fraction of a period remaining until the next reset date

- If there are 180 days in the coupon period and 57 days have passed since the last coupon, the Macaulay duration is
  - $MacDur_{Floating} = \frac{(180-57)}{180} = 0.683333$

- Floating-rate instruments typically have very low duration because coupon periods are typically less than six months in length. As a result, they are commonly used by investors to reduce duration in fixed-income portfolios.

View Markdown Source

```
### Macaulay duration for a floating-rate note or bond ($MACDur_{Floating}$)

$$
MacDur_{Floating} = \frac{(T - t)}{T} \tag{10}
$$

- As described in an earlier lesson, interest on floating-rate instruments
  varies depending on the level of a market reference rate ($MRR$) plus a
  quoted margin. At predetermined dates, payment amounts are reset to reflect
  changes in the $MRR$. Therefore, interest rate risk arises only between
  reset dates, because at the next reset date, coupon payments will adjust to
  the new $MRR$. Therefore, the Macaulay duration for a floating-rate note or
  bond is simply the fraction of a period remaining until the next reset date
- If there are $180$ days in the coupon period and $57$ days have passed since
  the last coupon, the Macaulay duration is
  - $MacDur_{Floating} = \frac{(180 - 57)}{180} = 0.683333$
- Floating-rate instruments typically have very low duration because coupon
  periods are typically less than six months in length. As a result, they are
  commonly used by investors to reduce duration in fixed-income portfolios.
```
