

Portfolio Mathematics

Quantitative Methods

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Learning Module 5: Portfolio Mathematics

Expected Return on Portfolio

$$E(R_P) = w_1E(R_1) + w_2E(R_2) + \dots + w_nE(R_n) \tag{1}$$

Where:

- $E(R_P)$: expected return on the portfolio
- w_i : proportion of portfolio invested in asset i
- $E(R_i)$: expected return on asset i
- n : number of assets in the portfolio

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## Expected Return on Portfolio
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$$
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E(R_{P})=w_{1} E(R_{1})+w_{2} E(R_{2})+\cdots+w_{n} E(R_{n}) \tag{1}
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$$
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Where:
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* $E(R_{P})$: expected return on the portfolio
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*  $w_i$ : proportion of portfolio invested in asset  $i$ 
*  $E(R_i)$ : expected return on asset  $i$ 
*  $n$ : number of assets in the portfolio
```

Portfolio Variance

$$\sigma^2(R_p) = E\{[R_p - E(R_p)]^2\} \quad (2)$$

Where:

- $\sigma^2(R_p)$: portfolio variance
- R_p : portfolio return
- $E(R_p)$: expected portfolio return

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## Portfolio Variance


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Where:

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*  $R_p$ : portfolio return
*  $E(R_p)$ : expected portfolio return
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Covariance of Returns

$$Cov(R_i, R_j) = E[(R_i - ER_i)(R_j - ER_j)] \quad (3)$$

Where:

- $Cov(R_i, R_j)$: covariance between returns on assets i and j
- R_i : return on asset i
- R_j : return on asset j

- $E(R_i)$: expected return on asset i
- $E(R_j)$: expected return on asset j
- Equation 3 states that the covariance between two random variables is the probability-weighted average of the cross-products of each random variable's deviation from its own expected value.

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Covariance of Returns

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Cov(R_{i}, R_{j}) =E[(R_{i}-ER_{i})(R_{j}-ER_{j})] \tag{3}
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```

Where:

- * $Cov(R_{i}, R_{j})$: covariance between returns on assets i and j
- * R_{i} : return on asset i
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- * $E(R_{i})$: expected return on asset i
- * $E(R_{j})$: expected return on asset j
- * Equation 3 states that the covariance between two random variables is the probability-weighted average of the cross-products of each random variable's deviation from its own expected value.

Sample Covariance of Returns

$$Cov(R_i, R_j) = \frac{\sum_{n=1}^n (R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j)}{(n - 1)} \tag{4}$$

Where:

- $Cov(R_i, R_j)$: sample covariance between returns on assets i and j
- $R_{i,t}$: return on asset i at time t
- $R_{j,t}$: return on asset j at time t
- \bar{R}_i : sample mean return on asset i
- \bar{R}_j : sample mean return on asset j
- n : sample of past data of size n

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Sample Covariance of Returns

\$\$

$$\text{Cov}(R_i, R_j) = \frac{\sum_{n=1}^n (R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j)}{(n - 1)}$$

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- * n : sample of past data of size n

Portfolio Variance Decomposition (Three-Asset Portfolio)

$$\begin{aligned}\sigma^2(R_p) &= E[(R_p - ER_p)^2] \\ &= E\{[w_1R_1 + w_2R_2 + w_3R_3 - E(w_1R_1 + w_2R_2 + w_3R_3)]^2\} \tag{5} \\ &= E\{[w_1R_1 + w_2R_2 + w_3R_3 - w_1ER_1 - w_2ER_2 - w_3ER_3]^2\} \\ &= w_1^2\sigma^2(R_1) + w_1w_2\text{Cov}(R_1, R_2) + w_1w_3\text{Cov}(R_1, R_3) \\ &\quad + w_1w_2\text{Cov}(R_1, R_2) + w_2^2\sigma^2(R_2) + w_2w_3\text{Cov}(R_2, R_3) \\ &\quad + w_1w_3\text{Cov}(R_1, R_3) + w_2w_3\text{Cov}(R_2, R_3) + w_3^2\sigma^2(R_3)\end{aligned}$$

The most compact way to state Equation 5 is

$$\sigma^2(R_p) = \sum_{i=1}^3 \sum_{j=1}^3 w_i w_j \text{Cov}(R_i, R_j)$$

Moreover, this expression generalizes for a portfolio of any size n to

$$\sigma^2(R_p) = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \text{Cov}(R_i, R_j) \tag{6}$$

Where:

- $\sigma^2(R_p)$: variance of portfolio return
- w_i : proportion of portfolio invested in asset i
- w_j : proportion of portfolio invested in asset j
- $\sigma^2(R_i)$: variance of return on asset i
- $\text{Cov}(R_i, R_j)$: covariance between returns on assets i and j

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## Portfolio Variance Decomposition (Three-Asset Portfolio)

$$
\sigma^2(R_p) = E[(R_p - ER_p)^2]
$$

$$
= E\{[w_1 R_1 + w_2 R_2 + w_3 R_3 - E(w_1 R_1 + w_2 R_2 + w_3 R_3)]^2\} \tag{5}
$$

$$
= E\{[w_1 R_1 + w_2 R_2 + w_3 R_3 - w_1 ER_1 - w_2 ER_2 - w_3 ER_3]^2\}
$$

$$
= w_1^2 \sigma^2(R_1) + w_1 w_2 \text{Cov}(R_1, R_2) + w_1 w_3 \text{Cov}(R_1, R_3)
$$

$$
+ w_1 w_2 \text{Cov}(R_1, R_2) + w_2^2 \sigma^2(R_2) + w_2 w_3 \text{Cov}(R_2, R_3)
$$
```

$$\sigma^2(R_p) = w_1^2 \sigma^2(R_1) + w_2^2 \sigma^2(R_2) + w_3^2 \sigma^2(R_3) + 2w_1w_2 \text{Cov}(R_1, R_2) + 2w_1w_3 \text{Cov}(R_1, R_3) + 2w_2w_3 \text{Cov}(R_2, R_3)$$

The most compact way to state Equation 5 is

$$\sigma^2(R_p) = \sum_{i=1}^3 \sum_{j=1}^3 w_i w_j \text{Cov}(R_i, R_j)$$

Moreover, this expression generalizes for a portfolio of any size n to

$$\sigma^2(R_p) = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \text{Cov}(R_i, R_j)$$

Where:

- * $\sigma^2(R_p)$: variance of portfolio return
- * w_i : proportion of portfolio invested in asset i
- * w_j : proportion of portfolio invested in asset j
- * $\sigma^2(R_i)$: variance of return on asset i
- * $\text{Cov}(R_i, R_j)$: covariance between returns on assets i and j

Correlation between two random variables

$$\rho(R_i, R_j) = \frac{\text{Cov}(R_i, R_j)}{[\sigma(R_i)\sigma(R_j)]} \quad (7)$$

Where:

- $\rho(R_i, R_j)$: correlation coefficient between returns on assets i and j
 - Alternative notations are $\text{Corr}(R_i, R_j)$ and p_{ij}
- $\text{Cov}(R_i, R_j)$: covariance between returns on assets i and j
- $\sigma(R_i)$: standard deviation of returns on asset i
- $\sigma(R_j)$: standard deviation of returns on asset j

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## Correlation between two random variables
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- * $\sigma(R_i)$: standard deviation of returns on asset i
- * $\sigma(R_j)$: standard deviation of returns on asset j

Covariance Given a Joint Probability Function

$$\text{Cov}(R_A, R_B) = \sum_i \sum_j P(R_{A,i}, R_{B,j})(R_{A,i} - ER_A)(R_{B,j} - ER_B) \quad (8)$$

Where:

- $\text{Cov}(R_A, R_B)$: covariance between random variables R_A and R_B
- $P(R_{A,i}, R_{B,j})$: joint probability of returns $R_{A,i}$ and $R_{B,j}$
- $R_{A,i}$: return i on asset A
- $R_{B,j}$: return j on asset B
- $E(R_A)$: expected return on asset A
- $E(R_B)$: expected return on asset B

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## Covariance Given a Joint Probability Function
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$$  
\operatorname{Cov}(R_{\{A\}}, R_{\{B\}}) = \sum_{\{i\}} \sum_{\{j\}} P(R_{\{A, i\}}, R_{\{B, j\}}) (R_{\{A, i\}} - ER_{\{A\}}) (R_{\{B, j\}} - ER_{\{B\}})  
$$
```

Where:

```
*  $\text{Cov}(R_A, R_B)$ : covariance between random variables  $R_A$  and  $R_B$ 
*  $P(R_{A, i}, R_{B, j})$ : joint probability of returns  $R_{A, i}$  and  $R_{B, j}$ 
*  $R_{A, i}$ : return  $i$  on asset  $A$ 
*  $R_{B, j}$ : return  $j$  on asset  $B$ 
*  $E(R_A)$ : expected return on asset  $A$ 
*  $E(R_B)$ : expected return on asset  $B$ 
```

Safety-First Ratio

$$\text{SFRatio} = \frac{E(R_P) - R_L}{\sigma_P} \tag{9}$$

Where:

- SFRatio: safety-first ratio
- $E(R_P)$: expected portfolio return
- R_L : threshold level (minimum acceptable return)
- σ_P : portfolio standard deviation

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## Safety-First Ratio


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Where:

*  $\text{SFRatio}$ : safety-first ratio
*  $E(R_P)$ : expected portfolio return
*  $R_L$ : threshold level (minimum acceptable return)
*  $\sigma_P$ : portfolio standard deviation
```