Probability Trees and Conditional Expectations

Quantitative Methods

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Learning Module 4: Probability Trees and Conditional Expectations

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Expected Value of a Discrete Random Variable X

$$E(X) = P(X_1)X_1 + P(X_2)X_2 + \ldots + P(X_n)X_n = \sum_{i=1}^{n} P(X_i)X_i$$

$$E(X) = \sum_{i=1}^{n} P(X_i)\,X_i \quad (1)$$

Where:

- $E(X)$: expected value of random variable X
- X_i: one of n possible outcomes of the discrete random variable X
- $P(X_i)$: probability of outcome X_i

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## Expected Value of a Discrete Random Variable X

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E(X) = P(X_1)X_1 + P(X_2)X_2 + \ldots + P(X_n)X_n = \sum_{i=1}^{n} P(X_i)X_i
$$
```

```
$$
E(X)=\sum_{i=1}^{n} P\left(X_{i}\right) X_{i} \tag{1}
$$

Where:

* $E(X)$: expected value of random variable $X$
* $X_i$: one of $n$ possible outcomes of the discrete random variable $X$
* $P(X_i)$: probability of outcome $X_i$
```

Variance of a Random Variable

$$\sigma^2(X) = E[X - E(X)]^2 \tag{2}$$

Where:

- $\sigma^2(X)$: variance of random variable X
- $E(X)$: expected value of X

The following equation summarizes the calculation of variance

$$\sigma^2(X) = P(X_1)[X_1 - E(X)]^2 + P(X_2)[X_2 - E(X)]^2 + ... +$$

$$+ ... + P(X_n)[X_n - E(X)]^2 = \sum_{i=1}^{n} P(X_i)[X_i - E(X)]^2 \tag{3}$$

Simplify

$$\sigma^2(X) = \sum_{i=1}^{n} P(X_i)[X_i - E(X)]^2$$

Where:

- $\sigma^2(X)$: variance of random variable X
- $E(X)$: expected value of X
- X_i: one of n possible outcomes of the discrete random variable X
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## Variance of a Random Variable

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Where:

* $\sigma^{2}(X)$: variance of random variable $X$
* $E(X)$: expected value of $X$

**The following equation summarizes the calculation of variance**

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\sigma^2(X) = P(X_1)[X_1 - E(X)]^2 + P(X_2)[X_2 - E(X)]^2 + \ldots +
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$$
+ \ldots + P(X_n)[X_n - E(X)]^2 = \sum_{i=1}^{n} P(X_i)[X_i - E(X)]^2 \tag{3}
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Simplify

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\sigma^2(X) =\sum_{i=1}^{n} P(X_i)[X_i-E(X)]^{2}
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Where:

* $\sigma^{2}(X)$: variance of random variable $X$
* $E(X)$: expected value of $X$
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Conditional Expected Value of a Random Variable

$$
E(X \mid S) = P(X_1 \mid S)X_1 + P(X_2 \mid S)X_2 + \cdots + P(X_n \mid S)X_n \tag{4}
$$

Where:

- $E(X \mid S)$: expected value of random variable X given event or scenario S
- X_i: one of n distinct outcomes $(X_1, X_2, \ldots, X_n)$
- $P(X_i \mid S)$: probability of outcome X_i given S

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## Conditional Expected Value of a Random Variable

$$
E(X \mid S)=P(X_1 \mid S) X_{1}+P(X_2 \mid S) X_{2}+\cdots+P(X_n \mid S) X_n \tag{4}
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Where:

* $E(X \mid S)$: expected value of random variable $X$ given event or scenario $S$
* $X_i$: one of $n$ distinct outcomes $(X_1, X_2, \ldots, X_n)$
* $P(X_i \mid S)$: probability of outcome $X_i$ given $S$
```

Total Probability Rule for Expected Value

$$E(X) = E(X \mid S)P(S) + E(X \mid S^{C})P(S^{C}) \tag{5}$$

Where:

- $E(X)$: unconditional expected value of X? (confirm!!!)
- $E(X \mid S)$: expected value of X given scenario S
- $P(S)$: probability of scenario S
- S^C: complement of S (event or scenario S does not occur)
- $P(S^C)$: probability of S^C

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## Total Probability Rule for Expected Value

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E(X) = E(X \mid S)P(S) + E(X \mid S^{C})P(S^{C}) \tag{5}
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Where:

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* $E(X \mid S)$: expected value of $X$ given scenario $S$
* $P(S)$: probability of scenario $S$
* $S^{C}$: complement of $S$ (event or scenario $S$ does not occur)
* $P(S^{C})$: probability of $S^{C}$
```

Total Probability Rule for Expected Value (General Case)

$$
E(X) = E(X \mid S_1)P(S_1) + E(X \mid S_2)P(S_2) + \cdots + E(X \mid S_n)P(S_n) \tag{6}
$$

Where:

- $E(X)$: unconditional expected value of X
- $E(X \mid S_i)$: expected value of X given scenario S_i
- $P(S_i)$: probability of scenario S_i
- $S_1, S_2, \ldots, S_n$: mutually exclusive and exhaustive scenarios or events

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## Total Probability Rule for Expected Value (General Case)

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E(X) = E(X \mid S_1)P(S_1) + E(X \mid S_2)P(S_2) + \cdots + E(X \mid S_n)P(S_n) \tag{6}
$$

Where:

* $E(X)$: unconditional expected value of $X$
* $E(X \mid S_i)$: expected value of $X$ given scenario $S_i$
* $P(S_i)$: probability of scenario $S_i$
* $S_{1}, S_{2}, \ldots, S_{n}$: mutually exclusive and exhaustive scenarios or events
```

Total Probability Rule

$$P(A)=\sum_{n} P(A \cap B_n) \tag{7}$$

Where:

* $P(A)$: unconditional probability of event A
* $P(A \cap B_n)$: probability of event A and event B_n occurring together
* B_n: event n in a set of mutually exclusive and exhaustive events

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## Total Probability Rule

$$
P(A)= \sum_{n} P(A \cap B_n) \tag{7}
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Where:

* $P(A)$: unconditional probability of event $A$
* $P(A \cap B_n)$: probability of event $A$ and event $B_n$ occurring together
* $B_n$: event $n$ in a set of mutually exclusive and exhaustive events
```

Bayes' Formula

$$= \frac{\text{Probability of the new information given event}}{\text{Unconditional probability of the new information}} \times \text{Prior probability of event.}$$

In probability notation, this formula can be written concisely as follows:

$$P(\text{ Event } \mid \text{ Information }) = \frac{P(\text{ Information } \mid \text{ Event })}{P(\text{ Information })} P(\text{ Event }) \tag{8}$$

or

$$P(A \mid B) = \frac{P(B \mid A)}{P(B)} \times P(A) \tag{8}$$

Where:

- $P(A \mid B)$: posterior probability of event A given information B
- $P(B \mid A)$: probability of observing information B given event A
- $P(A)$: prior probability of event A
- $P(B)$: unconditional probability of information B

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## Bayes' Formula

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P(A \mid B)=\frac{P(B \mid A)}{P(B)} \times P(A) \tag{8}
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Where:

* $P(A \mid B)$: posterior probability of event $A$ given information $B$
* $P(B \mid A)$: probability of observing information $B$ given event $A$
* $P(A)$: prior probability of event $A$
* $P(B)$: unconditional probability of information $B$
```
